Waves in plasmas :

Introduction :-

plasma is very much rich in wave phenomena. The study of waves in plasmas provides many information about the plasma properties. Generally two different approaches are used to study wave propagation in plasmas. The most straight forward and simpler approach is to solve plasma fluid equations simultaneously with Maxwell's equations and obtain a dispersion relation which relates wave frequency ω with wave number k. The plasma will be assumed to be homogeneous and of infinite extent so that there is no boundary effect.

2) Electron plasma wave in cold plasmas :-

To describe the above wave phenomena we starts with the following assumption —

ⓐ The plasma is assumed to be cold so that the plasma particles have no random thermal motion.

ⓑ The ions because of their heavier mass are at rest and form a uniform charge neutralizing background.

ⓒ The plasma is homogenous and infinitely extended

ⓓ There is no magnetic field.

ⓔ The electron motion occurs only along the x-direction.

Because of the assumption (e) we can write

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x}, \quad \vec{E} = E\hat{i}, \quad \vec{\nabla}\times\vec{E} = 0, \quad \therefore \vec{E} = -\vec{\nabla}\phi \qquad (1)$$

This implies that,

there is no fluctuating

magnetic field associated with the electron motion. So this is an electrostatic oscillation. We assume that the scale length of disturbance is sufficiently large and volume element over which the electric field $\vec{E}$ does not change significantly and contains significant number of electrons so that we may consider the electrons as a continuous fluid.

The fluid eqn of motion for the electron is given by —

$$m n_e \left(\frac{\partial}{\partial t} + \vec{u}_e \cdot \vec{\nabla}\right) \vec{u}_e = - e n_e \vec{E} \quad \text{—— (2)}$$

The equation of continuity is

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0 \quad \text{—— (3)}$$

The poisson's eqn for the electric field is —

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = e(n_i - n_e) \quad \text{—— (4)}$$

We assume that the ion density n_i remains constant at equillibrium value n_0. but the electron density slightly pertubed that is $n_e = n_0 + n_1$ where $n_1 << n_0$ —— (5)

Here the subscript zero suffers to equillibrium value and subscript one suffers to perturbation part.

For uniform nutral plasma at rest in the equillibrium state we have

$$\vec{\nabla} n_0 = \vec{u}_0 = \vec{E}_0 = 0$$

$$\frac{\partial n_0}{\partial t} = \frac{\partial \vec{u}_0}{\partial t} = \frac{\partial \vec{E}_0}{\partial t} = 0 \quad \text{—— (6)}$$

So eqn (2) to (4) in terms of perturbation quantities become

$$\frac{\partial \vec{u}_1}{\partial t} = -\frac{e}{m_e}\vec{E}_1 \quad \text{——— (7)}$$

$n_e = n_0$
$n_e = n_0 + n_1$

$$\frac{\partial n_1}{\partial t} + n_0 \vec{\nabla}\cdot\vec{u}_1 = 0 \quad \text{——— (8)}$$

$(u_1 \cdot \nabla) u_1$

$$\epsilon_0 \vec{\nabla}\cdot\vec{E}_1 = -e n_1 \quad \text{——— (9)}$$

Here we have assume the changes $n_1, \vec{u}_1, \vec{E}_1$ due to perturbation to be small quantities of 1st order and neglected higher order smaller terms considering one dimensional case and assuming that all the perturbation quantities vary as $e^{i(kx-\omega t)}$ and we get eqn (7) to (9)

$$-i\omega u_1 = -\frac{eE_1}{m_e} \quad \text{——— (10)}$$

$\frac{\partial}{\partial t} = -i\omega$
$\frac{\partial}{\partial x} = ik$

$$-i\omega n_1 + n_0 i k u_1 = 0 \quad \text{——— (11)}$$

$$\epsilon_0 i k E_1 = -e n_1 \quad \text{——— (12)}$$

now elliminating n_1 & u_1 we get —

$$u_1 = \frac{eE_1}{m_e \omega i} \qquad n_1 = \frac{\epsilon_0 i k E_1}{-e}$$

$$-i\omega\left[\frac{\epsilon_0 i k E_1}{-e}\right] + n_0 i k\left[\frac{eE_1}{m_e \omega i}\right] = 0$$

$$\Rightarrow \frac{\epsilon_0 k E_1 \omega}{e}\left[-1 + \frac{n_0 e^2}{m_e \omega^2 \epsilon_0}\right] = 0 \quad \text{——— (13)}$$

Since $E_1 \neq 0$ & we have

$$\frac{n_0 e^2}{m_e \omega^2 \epsilon_0} - 1 = 0$$

$$\Rightarrow \omega^2 = \frac{n_0 e^2}{m_e \epsilon_0} = \omega_{pe}^2 \text{ (say)} \quad \text{——— (14)}$$

ω_{pe} is known as electron plasma frequency.

The frequency of oscillation is const independent of the propagation const k. So that the group velocity $\frac{d\omega}{dK} = 0$

Group velocity determines the velocity with which energy propagates. Thus we conclude that the longitudinal electrostatic plasma oscillations do not propagate in a cold homogeneous plasma of infinite extent.

Representation of waves :-

Any periodic motion of a fluid can be recomposed by furier analysis in to a super position sinusoidal osscillations with different frequency ω and wave lengths λ.
A simple wave is any one of this components when the osscillation amplitude is small, the wave form is generally sinusoidal. Any sinusoidally osscillating quantity, say, the electric field $\vec{E}$ can be represented as

$$\vec{E} = E_0 \exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right] \quad (1)$$

when it Cartesian co-ordinates $\vec{k}\cdot\vec{r} = (k_x\hat{i} + k_y\hat{j} + k_z\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$

$= xk_x + yk_y + zk_z$

$\vec{k}$ is called propagation vector, ω be the frequency and E_0 be the const amplitude and the term $(\vec{k}\cdot\vec{r} - \omega t) = \theta$ is called the phase of the wave. If the wave propagates in the x-direction only then $\vec{k}$ has x component only and eqn (1) becomes $E = E_0 \exp[i(kx - \omega t)]$ —— (2)

By conversion the exponential notation means that the real part of the expressions is to be taken as the measurable quantity. Let us choose the const amplitude E_0 be real then the real part of E can be written as $Re(E) = E_0 \cos(kx - \omega t)$

A point of const phase on the wave —— (3) moves such that

$$\frac{d}{dt}(kx - \omega t) = 0$$

$$\Rightarrow k\frac{dx}{dt} - \omega = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{\omega}{k} = U_p\,(V_\phi)$$

This is called phase velocity of the wave. If ω/k is positive then wave moves to the right that is x-increases as t increases, so as to keep $(kx - \omega t)$ is const. If ω/k is negative then wave moves to the left.

Group velocity :-

The phase velocity of a wave in a plasma often exceeds the velocity of light c. This does not violet the theory of ~~negativity~~ relativity because an infinitely long wave of const amplitude can not carry any information

The carrier of a radio wave, for instance, carries no information until it is modulated. The modulation information does not travel at the phase velocity but at the group velocity which is always less than velocity of light c. Consider a modulated wave formed by adding and beating two waves of nearly ~~same for~~ equal frequencies. Let the waves be

$$E_1 = E_0 \cos\left[(k+\Delta k)x - (\omega+\Delta\omega)t\right]$$

$$E_2 = E_0 \cos\left[(k-\Delta k)x - (\omega-\Delta\omega)t\right]$$

E_1 & E_2 differ in frequency by ~~two~~ $2\Delta\omega$. Since each wave must have the phase velocity ω/k appropriate in the medium in which they propagate, one must allow for a difference $2\Delta k$ in propagation const. ~~Let~~

Let $a = kx - \omega t$ $\qquad a-b = (k-\Delta k)x - (\omega-\Delta\omega)t$

$$b = (\Delta k)x - (\Delta\omega)t$$

Then we have $E_1 + E_2 = E_0 \cos(a+b) + E_0 \cos(a-b)$

$$= 2E_0 \cos a \cos b$$

$$= 2E_0 \cos(kx-\omega t)\cos[(\Delta k)x - (\Delta\omega)t]$$

This is a sinusoidally modulated wave. The envelope of the wave given by $\cos[(\Delta k)x - (\Delta\omega)t]$ is what carries information

In travels at velocity $\delta\omega/\delta k$ taking the limit $\delta\omega \to 0$, we define the group velocity as $v_g = \frac{d\omega}{dk}$ it is this quantity that can not exceed C. Some general wave concepts before developing theory of waves in a plasma we recall some elementary result. For simplicity 1st consider electromagnetic waves propagate in vacuum. Then the electromagnetic eqns required that each component of $\vec{E}$ & $\vec{B}$ satisfied the wave equations.

$$\nabla^2 \psi = \frac{1}{C^2}\frac{\partial^2 \psi}{\partial t^2}$$

The plane wave solution is as follows.

$$\psi = A\exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right] + B\exp\left[-i\left(k\cdot r + \omega t\right)\right]$$

where A & B are const magnitude, k be the wave propagation vector and ω is the frequency choosing the x-axis along the wave vector $\vec{k}$.

A plane wave is one for which the wave disturbance is const over all points of a plane normal to the direction of propagation of a wave.

The general plane wave solution may be written by using Fourier analysis

$$\psi(x,t) = \int_{-\infty}^{\infty} A_k \exp\{ik(x-ct)\} + B_k \exp\{-ik(x+ct)\}$$

$$\Rightarrow \psi(x,t) = f(x-ct) + g(x+ct)$$

f & g are arbitrary func representing wave propagation from the left to right with velocity c.

polarization wave :-

In general to specify the electric field in a plane wave one must write a superposition of two linearly independent soln of the wave eqn as

$$E(x,t) = (E_y \hat{y} + E_z \hat{z}) \exp\{i(kx - \omega t)\} \quad \text{—— (1)}$$

in which E_y and E_z are complex amplitude and we take

$$E_y = E_{y0} e^{i\alpha} \qquad E_z = E_{z0} e^{i\beta}$$

where E_{y0} and E_{z0} are real.

then from (1)

$$\vec{E} = \left[E_{y0} \hat{y} + E_{z0} e^{i\delta} \hat{z} \right) \exp\{i(kx - \omega t + \alpha)\}$$

$$\text{where } \delta = \beta - \alpha \quad \text{—— (2)}$$

The electric vector at each point in a space rotates in a plane normal to x and it describes an ellipse. This is clearly seen for $\delta = \pm \frac{\pi}{2}$ Then eqn (2) can be rewritten as

$$\vec{E} = (E_{y0} \hat{y} \pm i E_{z0} \hat{z}) \exp\{i(kx - \omega t + \alpha)\} \quad \text{—— (3)}$$

By taking real part of eqn (3) we have

$$E_y(x,t) = E_{y0} \cos(kx - \omega t + \alpha)$$

and

$$E_z(x,t) = \mp E_{z0} \sin(kx - \omega t + \alpha)$$

which implies